PHY105 Final, Fall 06

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Problem 1

In this case we take coordinate $\hat{x}$ going up the slope, parallel to the slope and coordinate $\hat{y}$ perpendicular to it. $\hat{z}$ is coming out of the page.

There are three forces acting on this problem: gravitation (G), friction (f) and the normal (N) force from the plane. Since the second one is not a conservative force energy is not conserved in this set up.

Taking the slope in consideration the forces are:

$$F_G^x = -Mg\sin\theta \quad (1)$$
$$F_G^y = -Mg\cos\theta \quad (2)$$
$$F_f^x = -\mu N^y \quad (3)$$
$$F_f^y = 0 \quad (4)$$
$$N^x = 0 \quad (5)$$

Where we could set F_f^y to zero, because friction opposes the direction of motion and N^x to zero as the normal points perpendicular to the slope.

We can now write Newton's equations:

$$x: \quad M\ddot{x} = -Mg\sin\theta - \mu N^y \quad (6)$$
$$y: \quad M\ddot{y} = N^y - Mg\cos\theta = 0 \quad (7)$$

Where $\ddot{y} = 0$ as the sphere does not take off from the slope. This means we now know N^y and we can write the x equation as:

$$\ddot{x} = -g(\sin\theta + \mu\cos\theta) \quad (8)$$

Integrating this we obtain the velocity:

$$v = \dot{x} = v_0 - g(\sin\theta + \mu\cos\theta)t \quad (9)$$

Where we used the fact that the initial velocity was v_0.

Taking the CM of the sphere as the pivot, in this problem there is also a torque given by $\tau = \vec{r} \times \vec{F_f} = -R\hat{y} \times -\mu M g \cos\theta \hat{x} = -R\mu M g \cos\theta \hat{z}$, where $\vec{r}$ is the vector that joins the CM with the point of contact in the floor.

This gives an equation in the z direction for the rotation of the sphere:

$$I\ddot{\alpha} = -R\mu M g \cos\theta \tag{10}$$

The moment of inertia of a sphere is $I = \frac{2}{5}MR^2$. We can integrate this equation (using that the initial $\omega = \dot{\alpha} = 0$):

$$\omega = \dot{\alpha} = -\frac{5\mu g \cos\theta}{2R} t \tag{11}$$

The condition for no slipping is that the velocity of the point of contact should be zero. But the velocity of this point is related to the velocity of CM. We can then write an equation for time T when this condition is met:

$$0 = \vec{v} + \vec{\omega} \times \vec{r} = v\hat{x} + \omega\hat{z} \times (-R)\hat{y} = (v + \omega R)\hat{x} \tag{12}$$

If we plug in the values for this variables for $t = T$ we get:

$$v_0 - g(\sin\theta + \mu\cos\theta)T - R\frac{5\mu g \cos\theta}{2R}T = v_0 - g\left[\sin\theta + \frac{7}{2}\mu\cos\theta\right]T \tag{13}$$

Therefore:

$$T = \frac{v_0}{g\left[\sin\theta + \frac{7}{2}\mu\cos\theta\right]} \tag{14}$$

This expression has the correct limiting values as μ becomes very large or small or as the angle approaches 0 or $\frac{\pi}{2}$.

Problem 2

2.a)

We first calculate the CM of the system just after the collision. We choose a system of coordinates centered at the center of the ring and such that the coordinates of the particle that just collided are $\vec{R_p} = (x_p, y_p) = (0, -R) = 0\hat{x} - R\hat{y}$. Let's calculate the CM as:

$$\vec{R_{CM}} = \frac{\int \vec{r}dm + \vec{R_p}M}{M + M} = \frac{1}{2}\frac{\int \vec{r}dm}{M} + \frac{1}{2}\vec{R_p} \tag{15}$$

Where the integral is over the ring. This means the CM position can just be calculated as the center of mass of the particle and a new particle with the same mass located at the center of the ring (this is the CM of the ring). Then:

$$\vec{R_{CM}} = \frac{1}{2}\left[0\hat{x} + 0\hat{y} + 0\hat{x} - R\hat{y}\right] = 0\hat{x} - \frac{R}{2}\hat{y} = (0, -\frac{R}{2}) \tag{16}$$

2.b)

We want to calculate I_{CM} now. Let us start with the simpler calculation of I_0, the moment of inertia around the center of coordinates:

$$I_0 = \int r^2 dm + R^2 M = 2MR^2 \tag{17}$$

Because of the parallel axis theorem we know that the moment of inertia around any parallel axis is related to the CM moment of inertia as $I_0 = I_{CM} + M_T d^2$, where M_T is the total mass of the system and d is the distance between axis. In our case $M_T = 2M$ and $d = \frac{R}{2}$. This yields:

$$I_{CM} = I_0 - 2M\left(\frac{R}{2}\right)^2 = 2MR^2 - \frac{1}{2}MR^2 = \frac{3}{2}MR^2 \tag{18}$$

Note that $I_0 > I_{CM}$, as it should be.

2.c)

Energy is not conserved in this case as the collision is not elastic. We can use conservation of angular momentum, instead. Around the CM the initial angular momentum is (always in the $\hat{z}$ direction) $L_0 = \frac{R}{2}Mv$. Notice the $\frac{1}{2}$ factor. This comes about because we are calculating L_0 around the CM, not the center of the ring. Also, it is clear that only the particle carries angular momentum initially as the ring is at rest. The final angular momentum is given by $L_f = I_{CM}\omega$. We now know I_{CM}, then:

$$\frac{R}{2}Mv = \frac{3}{2}MR^2\omega \quad \rightarrow \quad \omega = \frac{v}{3R} \tag{19}$$

Problem 3

3a. The new equilibrium position is that place where the spring force (from Hooke's Law) balances the weight Mg of the mass. Thus

$$-k(\Delta y) = Mg$$

and

$$\Delta y = -Mg/k$$

3b. Energy is conserved here. The kinetic energy is zero both at the moment of the drop and at the moment of maximum compression. Energy is merely exchanged between spring potential energy and gravitational potential energy.

Let gravitational potential be defined to equal zero at the point $y = 0$ (the undisturbed height of the spring):

$$\begin{aligned} E_{\rm drop} &= E_{\rm max} \\ Mgh &= Mgy_{\rm max} + \frac{1}{2}ky_{\rm max}^2 \end{aligned}$$

This is a quadratic equation in the unknown $y_{\rm max}$, to which we can apply the usual quadratic formula to find the two solutions:

$$\begin{aligned} y_{\rm max} &= -\frac{Mg}{k} \pm \frac{\sqrt{(Mg)^2 - 4(k/2)(-Mgh)}}{k} \\ &= -\frac{Mg}{k} \pm \frac{\sqrt{(Mg)^2 + 2kMgh}}{k} \end{aligned}$$

Here we clearly want the negative sign, as the other choice is a positive value, corresponding to the maximum *extension* of the spring. The answer can be written in at least two sensible forms:

$$\begin{aligned} y_{\rm max} &= -\frac{Mg}{k} - \frac{\sqrt{(Mg)^2 + 2kMgh}}{k} \\ &= -\frac{Mg}{k}\left(1 + \sqrt{1 + \frac{2kh}{Mg}}\right) \end{aligned}$$

3c. We know that this system is a simple harmonic oscillator, because the net force on the mass due to the spring plus gravity increases linearly with distance from the equilibrium position we called Δy in part (a). That is,

$$\begin{aligned} F_{\rm net} &= Ma \\ -Mg - k(y - \Delta y) &= M\ddot{y} \\ -Mg - k(y + Mg/k) &= M\ddot{y} \\ -ky &= M\ddot{y} \end{aligned}$$

This equation is in the form of the simple harmonic oscillator $\ddot{y} + \omega^2 y = 0$, from which we read off that $\omega = \sqrt{k/M}$ and

$$T = 2\pi/\omega = 2\pi\sqrt{M/k}$$

The answer to part (b) is clearly a turning point in the oscillation, while part (a) gives the center of the oscillation (we can recognize this, because it's where $F_{\rm net} = 0$ or $\ddot{y} = 0$). Thus the amplitude A is

$$A = |y_{\rm max} - \Delta y| = \frac{Mg}{k}\sqrt{1 + \frac{2kh}{Mg}}$$

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Problem 4

4a. If the disk accelerates in the θ direction then the corresponding torque has magnitude $I_r\ddot{\theta}$ and is in the B to A direction. You all know $I_s = mr^2/2$. To find I_r look at the moment of inertia tensor on the formula sheet and note that $I_{xx} = I_{yy} = I_{zz}/2$. Thus, $\tau = I_s\ddot{\theta}/2$.
4b. The component of angular momentum in the plane of the turntable is $L_s \sin\theta$. The time rate of change of this as the table rotates is $L_s\Omega\sin\theta$ as you have shown many times. Here $L_s = I_s\omega_s$.
4c. The net torque must be zero since the spinning disk is unconstrained along the AB axis. Thus

$$\begin{aligned} I_s\ddot{\theta}/2 + I_s\omega_s\Omega\sin\theta &= 0 & (20)\\ \ddot{\theta} + 2\omega_s\Omega\sin\theta &= 0 & (21)\\ \ddot{\theta} + 2\omega_s\Omega\theta &= 0 & (22)\end{aligned}$$

where the last line is the small angle approximation.
4d. From the bottom above, the oscillation frequency is $\sqrt{2\omega_s\Omega}$
4e. The disk oscillates about the z-axis (directed along $\vec{\Omega}$). If there is damping then the amplitude of oscillation decreases with time. After many time constants, the two spin axes become parallel. This is a neat and general phenomena. If the earth is the rotating table, ω_s points north. The device is discussed in K&K on pages 302-304.

Problem 5

a) We need to integrate the force to get the potential:

$$U(r) = -\int_0^r -kr^n\hat{\vec{r}} \cdot (\hat{\vec{r}}\,dr) = +k\int_0^r r^n\,dr = \frac{1}{n+1}kr^{n+1}\,.$$

b) The total energy is constant. The energy consists of the potential energy worked out in part (a), and the kinetic energy which has terms due to the velocity component in the radial direction and in the azimuthal direction. (The orbit is confined to a plane perpendicular to the direction of the angular momentum which is constant.) So

$$E = m\dot{r}^2/2 + mr^2\dot{\theta}^2/2 + U(r)\,.$$

We use the fact that $\ell = mr^2\dot{\theta}$ to eliminate the dependence on $\dot{\theta}$ and obtain an expression for the energy that depends only on r, $\dot{r}$ and the constant angular momentum, ℓ:

$$E = m\dot{r}^2/2 + \ell^2/(2mr^2) + kr^{n+1}/(n+1)\,,$$

and the radial motion of this particle is the same as the motion of a particle moving in one dimension with a force produced by the effective potential:

$$U_{\text{eff}} = \ell^2/(2mr^2) + kr^{n+1}/(n+1)\,.$$

c) The period of the orbit is $T = 2\pi/\dot{\theta}$. Further, $\dot{\theta} = \ell/mr^2$, so the period is $2\pi mr^2/\ell$. We need to eliminate r. Since the orbit is circular, r is constant and the particle is located at the minimum of the effective potential. To find the minimum, we set the derivative of the effective potential to 0:

$$0 = -\ell^2/(mr^3) + kr^n \, .$$

Solve for r

$$r = \left(\frac{\ell^2}{km}\right)^{1/(n+3)} \, .$$

Now we can plug into the expression for the period (being very careful with the arithmetic for the exponents!):

$$T = 2\pi \frac{m}{\ell} \left(\frac{\ell^2}{km}\right)^{2/(n+3)} = 2\pi \left(\frac{m^{n+1}}{k^2 \ell^{n-1}}\right)^{1/(n+3)} \, .$$

d) Again, we use the concept of a 1D motion in the effective potential. The 2D motion of the particle in a circular orbit corresponds to 1D motion where the particle is at rest at the equilibrium point of the effective potential. Since the particle is given a perturbation in the radial direction, the angular momentum is unchanged. This means the 1D particle moves in the same effective potential and is oscillating about the minimum. For small oscillations, the effective potential must be a quadratic function of the distance from the equilibrium point. So, we differentiate the effective potential twice to find the effective spring constant, k_{eff}. Then, $T_p = 2\pi\sqrt{m/k_{\text{eff}}}$. Since we've already differentiated the effective potential once in part (c), we start from there and differentiate again:

$$k_{\text{eff}} = 3\ell^2/(mr^4) + nkr^{n-1} \, .$$

We need to evaluate this at the equilibrium radius (again from part (c)):

$$k_{\text{eff}} = 3\frac{\ell^2}{m} \left(\frac{\ell^2}{km}\right)^{-4/(n+3)} + nk \left(\frac{\ell^2}{km}\right)^{(n-1)/(n+3)} \, .$$

This can be simplified (again being careful with the exponent arithmetic!):

$$k_{\text{eff}} = (3+n) k^{4/(n+3)} \ell^{2(n-1)/(n+3)} m^{-(n-1)/(n+3)} \, .$$

Now we plug this into the expression for T_p and simplify:

$$T_p = 2\pi \frac{1}{\sqrt{n+3}} \left(\frac{\ell^2}{km}\right)^{2/(n+3)} = 2\pi \frac{1}{\sqrt{n+3}} \left(\frac{m^{n+1}}{k^2 \ell^{n-1}}\right)^{1/(n+3)} \, .$$

Dividing by the expression for T from part (c),

$$\frac{T_p}{T} = \frac{1}{\sqrt{n+3}} \, ,$$

an amazingly simple result! Since we want $T_p = T/3$, we need $n = 6$.

Problem 6

6a. The adiabat is the 1-3 process. Recall that an isotherm has $P \propto V^{-1}$ and an adiabat has $P \propto V^{-\gamma}$, where $\gamma = 5/3$ for a monatomic gas. Since $\gamma > 1$, the adiabat is steeper.
6b. It is easiest to follow the adiabatic process.

$$P_3 V_3^\gamma = P V^\gamma \tag{23}$$
$$P_3 = (V/V_3)^\gamma P \tag{24}$$
$$P_3 = (1/2)^\gamma P \tag{25}$$

6c. There are many approaches to determining the efficiency. Perhaps the most direct is:

$$\eta = W_{by}/Q_{in} \tag{26}$$
$$\eta = (Q_{in} - Q_{out})/Q_{in} \tag{27}$$
$$\eta = 1 - Q_{out}/Q_{in} \tag{28}$$

Heat flows in on the isotherm and out on the isochoric process. For the isotherm, $Q_{in} = nRT_1 \ln 2$. For the isochor:

$$Q_{out} = C_V(T_2 - T_3) \tag{29}$$
$$= C_V(T_1 - T_3) \tag{30}$$
$$= C_V(T_1 - \frac{P_3 V_3}{nR}) \tag{31}$$
$$= C_V(T_1 - \frac{2PV}{2^\gamma nR}) \tag{32}$$
$$= C_V T_1 (1 - \frac{2}{2^\gamma}) \tag{33}$$

and so

$$\eta = 1 - Q_{out}/Q_{in} \tag{34}$$
$$\eta = 1 - \frac{3}{2\ln 2}(1 - \frac{2}{2^\gamma}) \tag{35}$$
$$\eta = 20\% (not required) \tag{36}$$

Problem 7

a) By momentum conservation, the momentum of the fragments must be equal and opposite. Since each fragment has the same magnitude of momentum and the same rest mass, each fragment must have the same energy. By energy conservation the energy of each fragment is $M_0c^2/2$.

b) We know the energy and rest mass of each fragment and the ratio is just γ. So

$$\gamma = (M_0c^2/2)/(2M_0c^2/5) = 5/4\,.$$

Or

$$16/25 = 1/\gamma^2 = 1 - v^2/c^2\,,$$

which gives

$$v = 3c/5\,.$$

c) $p^2c^2 = E^2 - m^2c^4$, where $m = (2/5)M_0$ is the rest mass of each fragment. So,

$$p^2 = (1/4)M_0^2c^2 - (4/25)M_0^2c^2 = (9/100)M_0^2c^2\,.$$

and

$$p = (3/10)M_0c\,.$$

We could also have used the expression $p = \gamma mv$:

$$p = (5/4)((2/5)M_0)((3/5)c) = (3/10)M_0c\,.$$