Solutions to 105 Midterm

Ed Groth, 23-Oct-2006

1. Let $\boldsymbol{F}$ be the lift force on the plane. This is applied at angle θ_b from the vertical. The sum of the forces must equal the mass times the acceleration. In the vertical direction we have $F\cos\theta_b - mg = 0$ or $F\cos\theta_b = mg$. In the horizontal direction, $F\sin\theta_b = mv_0^2/r$. Divide one by the other to find $\tan\theta_b = v_0^2/gr$ or

$$r = \boxed{\frac{v_0^2}{g\tan\theta_b}} .$$

2. The thrust is $F_t = -u(dM/dt)$ If you don't remember this you should derive the rocket equation. Let $v(t)$ be the velocity of the rocket (relative to the ground). Let u be the velocity of the exhaust relative to the rocket. (So u is negative.) Let $M(t)$ be the mass of the rocket and let $dM(t)/dt$ be the rate of change of the rocket mass (because fuel is being expelled). Consider the interval $t \longrightarrow t + dt$. Conservation of momentum tells us that

$$M(t)v(t) = (M + dM)(v + dv) + (-dM)(v + u + dv) = Mv + M\,dv - u\,dM .$$

Or

$$M\frac{dv}{dt} = u\frac{dM}{dt} = \text{Thrust} .$$

In what follows, we'll take u to be a positive number, which means a minus sign is introduced and the thrust is given by the expression above.

(a) The thrust must equal the weight. So

$$-u\frac{dM}{dt} = Mg_m \qquad \text{or} \qquad -\frac{dM}{dt} = \boxed{\frac{Mg_m}{u}} .$$

(b) As the mass decreases, the thrust must decrease so it's always equal to the weight:

$$-u\frac{dM}{dt} = Mg_m \qquad \text{or} \qquad \frac{dM}{M} = -\frac{g_m}{u}\,dt .$$

Integrating,

$$\ln\frac{M_f}{M_0} = -\frac{g_m}{u}t .$$

The final mass is $(2/3)M_0$, so

$$t_{\text{hover}} = \boxed{\frac{u}{g_m}\ln\frac{3}{2}} .$$

3.

(a)

$$K = \frac{1}{2} I\omega^2 = \boxed{\frac{1}{24} M\ell^2\omega^2} \; .$$

(b) Conservation of linear momentum implies $0 = Mv + mu$ or $u = -(M/m)v$. Conservation of energy gives

$$\frac{1}{24} M\ell^2\omega^2 = \frac{1}{2} Mv^2 + \frac{1}{2} mu^2 \; .$$

We substitute for u in the energy equation and solve for v (which will be negative if we take the positive direction to the right):

$$v = \boxed{-\ell\omega\sqrt{\frac{1}{12(1 + M/m)}}} \; .$$

We multiply by $-(M/m)$ to get u:

$$u = \boxed{\ell\omega\frac{M}{m}\sqrt{\frac{1}{12(1 + M/m)}}} \; .$$

(c) Here we finally get to use conservation of angular momentum! The angular momentum of the rod about its center is $I\omega$. After the collision, the rod has no angular momentum about its original center, but the ball does. It's mux. So

$$x = \frac{I\omega}{mu} = \boxed{\ell\sqrt{\frac{1 + M/m}{12}}} \; .$$

Note that the $x \le \ell/2$, so $M/m \le 2$.

4. The mass is given by $m = \int_0^\ell \rho_0 (x/\ell)^3\,dx = \rho_0\ell/4$.

(a) The center of mass is the mass weighted average position.

$$x_{\rm cm} = \frac{1}{\rho_0\ell/4}\int_0^\ell \rho_0\left(\frac{x}{\ell}\right)^3 x\,dx = \frac{1}{\rho_0\ell/4}\frac{\rho_0\ell^2}{5} = \boxed{\frac{4}{5}\ell}\ .$$

(b) Set the sum of the torques about the A end of the rod to zero: $T_{\rm B}\ell - mg(4\ell/5) = 0$, so

$$T_{\rm B} = \boxed{\frac{4}{5}mg}\ .$$

(c) Now the sum of the torques about the A end of the rod is the moment of inertia (with A as center) times the angular acceleration. The first step is to calculate the moment of inertia:

$$I_{\rm A} = \frac{m}{\rho_0\ell/4}\int_0^\ell \rho_0\left(\frac{x}{\ell}\right)^3 x^2\,dx = \frac{m}{\rho_0\ell/4}\frac{\rho_0\ell^3}{6} = \frac{2}{3}m\ell^2\ .$$

Then $I\alpha = mg(4\ell/5)$ and

$$\alpha = \boxed{\frac{6}{5}\frac{g}{l}}\ .$$

If we take the A end on the left and the B end on the right, this is a clockwise angular acceleration.

(d) The magnitude of the linear acceleration of the center of mass is the angular acceleration times $x_{\rm cm}$. This gives $a = (4\ell/5)\cdot(6g/5\ell) = 24g/25$. Of course this is downward. The tension in string A plus the weight gives the vertical acceleration times the mass. So

$$T_{\rm A} = \boxed{\frac{1}{25}mg}\ .$$