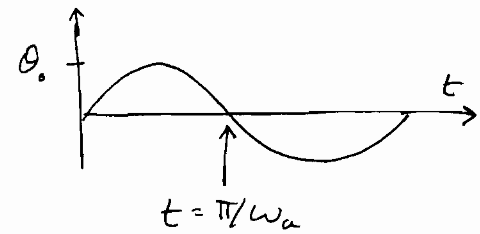
PHY105 MT2 Solutions (Revised) Dec 9 '05 Exam

1 a) $I\ddot{\theta} = -C\theta \Rightarrow \ddot{\theta} + \frac{C}{I}\theta = 0 \Rightarrow \omega_a^2 = 2C/MR^2$

b) With $\theta(t) = \theta_0 \sin(\omega_a t)$ motion is



Disk is moving with angular velocity and passing through the origin. Adding a ring of putty concentrically does not change L: $I' = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2 \Rightarrow \omega_b^2 = 2C/3MR^2$.

c) Angular momentum is conserved (not energy: "collision" is inelastic) $\Rightarrow I_a\theta_a\omega_a = I_b\omega_b\theta_b$

$$\Rightarrow \theta_b = \frac{I_a\omega_a}{I_b\omega_b}\theta_0 = \frac{\frac{1}{2}MR^2(2C/MR^2)^{1/2}}{\frac{3}{2}MR^2(2C/3MR^2)^{1/2}}\theta_0 = \frac{\sqrt{3}}{3}\theta_0 = \frac{\theta_0}{\sqrt{3}}$$

2a) Circular orbit $\frac{mv_0^2}{R} = \frac{GmM_E}{R^2} \Rightarrow \frac{1}{2}mv_0^2 = \frac{GmM_E}{2R} \Rightarrow E = E_k + E_G = -\frac{GM_E m}{2R}$

$T_0 = \frac{2\pi R}{v_0}$, $v_0 = \left(\frac{GM_E}{R}\right)^{1/2}$ from above, $\Rightarrow T_0 = 2\pi\frac{R^{3/2}}{(GM)^{1/2}}$

or $A = 2R$. $\ell = Rmv_0 = m(GM_E R)^{1/2}$

b) Extended solution (we did not expect you to have all of this). The mass of the new piece is $m_A = m/4$. Since $\vec{v}_A = 0$, it's angular momentum is 0. Let's examine an orbit in the limit as $\ell \to 0$ but with $E = -\frac{GM_E m_A}{R}$. $\varepsilon^2 = 1 + \underbrace{\frac{2E\ell^2}{\mu c^2}}_{-2\alpha} = 1 - 2\alpha$ when α is small*. Now

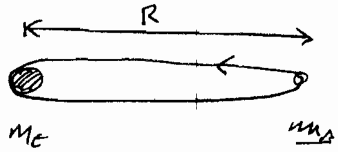
$-2\alpha = \frac{2\ell^2}{\mu c^2}\left(-\frac{GM_E m_A}{R}\right)$. For $M_E \gg m_A$, $\mu = m_A \Rightarrow \alpha = \frac{GM_E\ell^2}{Rc^2}$

* (High eccentricity $\Rightarrow \varepsilon \to 1$)

The parameter $r_0 = \frac{\ell^2}{m_A C}$ also goes to 0 as $\ell \to 0$. We

see that $\alpha = \frac{G M_E m_A}{R C} \frac{\ell^2}{m_A C} = \frac{G M_E m_A}{R C} r_0 = \frac{r_0}{R} \Rightarrow r_0 = \alpha R$

Thus the major axis is $A = \frac{2 r_0}{1-\varepsilon^2} \to \frac{2\alpha R}{2\alpha} \to R$ and

the orbit is 

as suggested in the hint.

Then, using $T = \left(\frac{\pi^2 A^3}{2(M_E + m_A) G}\right)^{1/2}$ from the formula sheet, $t_{fall} = \frac{1}{2}\left(\frac{\pi^2 R^3}{2 M_E G}\right)^{1/2}$

$= \frac{\pi}{2\sqrt{2}} \frac{R^{3/2}}{(M_E G)^{1/2}} = \frac{\pi}{2\sqrt{2}} \frac{T_0}{2\pi} \Rightarrow t_{fall} = T_0 / 4\sqrt{2}$

We could also have done this by integrating:

$\frac{1}{2} m_A v^2 - \frac{G M_E m_A}{r} = -\frac{G M m_A}{R} \Rightarrow \left(\frac{dr}{dt}\right)^2 = 2 G M_E \left(\frac{1}{r} - \frac{1}{R}\right) = \underbrace{\frac{2 G M_E}{R}}_{2 v_0^2} \left(\frac{R-r}{r}\right)$

$\Rightarrow \int_0^{t_{fall}} dt = \int_R^0 \frac{1}{\sqrt{2} v_0} \left(\frac{r}{R-r}\right)^{1/2} dr$ But $\int \left(\frac{p+x}{q-x}\right)^{1/2} = -(p+x)^{1/2}(q-x)^{1/2} - (p+q)\sin^{-1}\left(\frac{q-x}{p+q}\right)$ $p=0$, $q=R$

(we didn't expect you to know this!). Plugging in

$\sqrt{2} v_0 t_{fall} = \underbrace{0 - R\sin^{-1}\left(\frac{R}{R}\right)}_{r=0} - \underbrace{\left[0 - R \sin^{-1} 0\right]}_{r=R} = -R\frac{\pi}{2} + R\pi = R\frac{\pi}{2}$

$\Rightarrow t_{fall} = \frac{\pi R}{2\sqrt{2} v_0} = \frac{\cancel{\pi} \cancel{R}}{2\sqrt{2}} \frac{T_0}{2\cancel{\pi} R} = \frac{T_0}{4\sqrt{2}}$, same as before.

2c) ℓ is conserved, K is added. By consv of p in the "explosion" $m v_0 = \frac{m}{4} v_A + \frac{3}{4} m v_B \Rightarrow v_B = \frac{4}{3} v_0$ $\quad m_B = \frac{3}{4} m = \mu_B$

$E = \frac{\ell^2}{2\mu_B R^2} - \frac{G M_E m_B}{R}$ $(\dot{r} = 0) \Rightarrow E = \frac{m^2 G M_E R}{2 \frac{3}{4} m R^2} - \frac{3}{4} \frac{G M_E m}{R}$

$= -\frac{1}{12} \frac{G M_E m}{R} = -\frac{1}{9} \frac{G M_E m_B}{R} \Rightarrow A = -\frac{G M_E m_B}{E} = \frac{-G M_E m_B}{-\frac{1}{9} \frac{G M_E m_B}{R}} = 9R$

$\Rightarrow a = 4.5R$. Likewise $\varepsilon = 7/9$

3a) $y(x,t) = \frac{1}{2}\cos(2x-6t)$ by $V = \omega/k$ all MKS or SI units

b) $dy/dt = 3\sin(2x-6t) \Rightarrow$ at $x=1m$; $t=2sec$ $y = \frac{1}{2}\cos(-10)$

and $dy/dt = -3\sin 10 = 1.6$ m/sec up.

c) Easiest to do graphically. The wave moves to the left at 2m/sec and so the bit at the origin moves up 1 m in 1 sec, or 1m/sec. The bit at x=1 has been moving up at 1m/sec for ½ sec. etc.