Solutions to 105 Midterm 2

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1.

(a) The gravitational force and the normal force act through the center of mass and do not make a torque about the center of mass. The friction force acts at the edge of the ring, perpendicular to the plane of the ring, so it makes a torque whose magnitude is $\boxed{fb}$ and whose direction, at the instant shown in the figure in the exam, is in the plane of the ring, $\boxed{\text{out of the paper}}$.

(b) The torque computed in part (a) is the rate of change of the (horizontal) component of angular momentum of the ring:

$$fb = \frac{dL_h}{dt} .$$

The angular momentum of the ring (for the rotation about its axis) points towards the apex of the cone (parallel to the side of the cone). As the ring goes around, the vertical component of the angular momentum is constant and the horizontal component has constant magnitude but rotates with angular frequency Ω. So,

$$\frac{1}{L_h}\frac{dL_h}{dt} = \Omega .$$

We know the time derivative, but what is L_h? If the ring is rotating about its axis at angular speed ω, then $L = mb^2\omega$. Ω and ω are related since

$$\Omega = \frac{v}{R} = \frac{v}{h\tan\theta} , \tag{1}$$

and

$$\omega = \frac{v}{b} .$$

If this is hard to see, note that the number of meters laid down by the rim in time dt is $\omega b\,dt$ and the number of meters traversed on the cone in the same time is $\Omega R\,dt$. So $\omega = R\Omega/b$ and

$$\frac{1}{mRb\Omega\sin\theta} fb = \Omega ,$$

or

$$f = \boxed{mh\Omega^2\tan\theta\sin\theta} . \tag{2}$$

3. (continued)

The ring is also instantaneously rotating about its diameter perpendicular to the surface of the cone. The angular speed and angular momentum of this rotation are proportional Ω. To the extent that $b \ll R = h \tan\theta$ as indicated by the figure, this component of angular momentum is much smaller than the spin angular momentum and it has been ignored (the gyroscope approximation!).

(c) In addition to the torque equation, we have the vertical and horizontal force equations. In the vertical direction, there is no acceleration and

$$f \cos\theta + N \sin\theta = mg \; . \tag{3}$$

In the horizontal direction, the normal force acts inward and the friction force acts outward. Together, they combine to provide the centripetal acceleration.

$$N \cos\theta - f \sin\theta = \frac{mv^2}{R} = \frac{mv^2}{h \tan\theta} \; , \tag{4}$$

since we can ignore the small distance ($b \cos\theta$) by which the center of mass is inside the circle traced by the point of contact.

Equations (1) and (2) give v and f. We plug these into equations (3) and (4), eliminate N and solve for Ω.

$$mh\Omega^2 \tan\theta \sin\theta \cos\theta + N \sin\theta = mg \; ,$$

$$N = \frac{mg - mh\Omega^2 \sin^2\theta}{\sin\theta} \; .$$

$$\frac{mg - mh\Omega^2 \sin^2\theta}{\sin\theta} \cos\theta - \left(mh\Omega^2 \tan\theta \sin\theta\right) \sin\theta = mh\Omega^2 \tan\theta \; .$$

Note that we can factor out a $\cos^2\theta + \sin^2\theta = 1$ from the Ω^2 term on the left hand side:

$$\frac{mg}{\tan\theta} - mh\Omega^2 \tan\theta = mh\Omega^2 \tan\theta \; ,$$

or

$$\Omega^2 = \frac{g}{2h \tan^2\theta} \; .$$

$$\Omega = \boxed{\sqrt{\frac{g}{2h \tan^2\theta}}} \; .$$

2. If the velocity far away was 0, then the energy is 0. Also, the mass of the comet is so small compared to the mass of the Earth, that we can regard the Earth as fixed.

(a) At $R = 100R_e$,

$$0 = -\frac{GM}{R} + \frac{1}{2}v_r^2 ,$$

where we have divided out the mass of the comet. So,

$$v_r = \sqrt{\frac{2GM}{R}} = \boxed{\sqrt{\frac{2GM}{100R_e}}} .$$

(b) By giving a perpendicular velocity $v_\perp$, we are giving some energy $v_\perp^2/2$ (per unit mass) and some angular momentum $v_\perp R$ (per unit mass). When the comet just grazes the surface of the Earth at its perigee it has only a tangential velocity, v_p and angular momentum per unit mass (conserved) is $v_p R_e = v_\perp R$. Also energy is conserved, so (per unit mass),

$$-\frac{GM}{R_e} + \frac{1}{2}v_p^2 = \frac{1}{2}v_\perp^2 .$$

We use the angular momentum equation to solve for v_p and plug that into the energy equation to solve for $v_\perp$. The result is

$$v_\perp = \sqrt{\frac{2GMR_e}{(100R_e)^2 - R_e^2}} = \boxed{\sqrt{\frac{2GM}{9999R_e}}} \approx \frac{v_r}{10} .$$

3.

(a) When the pendulum is displaced to the side by angle θ, the restoring torque is $\tau = -Mg\ell \sin\theta \approx -Mg\ell\theta$. We set this equal to the moment of inertia times the angular acceleration or $I\ddot{\theta}$. The moment of inertia is found from the parallel axis theorem. The moment of inertia about the center of mass is $2MR^2/5$ and that of a point mass at distance ℓ from the pivot is $M\ell^2$. The total is $I = M(\ell^2 + 2R^2/5)$. So

$$M(\ell^2 + 2R^2/5)\ddot{\theta} = -Mg\ell\theta \,.$$

or

$$\ddot{\theta} + \frac{g}{\ell(1 + 2R^2/5\ell^2)}\,\theta = 0 \,,$$

from which we read off:

$$\omega_0 = \boxed{\sqrt{\frac{g}{\ell(1 + 2R^2/5\ell^2)}}} \,.$$

(b) Ignoring any buoyant force, when the sphere has reached terminal velocity, the drag force and the gravitational force are equal. We expect the drag force to be proportional to the velocity, so

$$Mg = bv_t \quad \text{or} \quad b = Mg/v_t \,.$$

The velocity of the sphere when it's oscillating is $\ell\dot{\theta}$. So the drag force is $b\ell\dot{\theta}$ with b given above. Then the drag torque is $-b\ell^2\dot{\theta}$ and the equation of motion of the pendulum for small oscillations becomes

$$I\ddot{\theta} = -b\ell^2\dot{\theta} - Mg\ell\theta \,.$$

Plugging in for I and b and rearranging, we have

$$\ddot{\theta} + \frac{g}{v_t(1 + 2R^2/5\ell^2)}\,\dot{\theta} + \frac{g}{\ell(1 + 2R^2/5\ell^2)}\,\theta = 0 \,,$$

from which we read off:

$$\gamma = \boxed{\frac{g}{v_t(1 + 2R^2/5\ell^2)}} \,.$$

3. (continued)

(c) The easiest way to "solve" this differential equation is to guess a solution and see if it works. Since we are looking for damping and oscillations, we expect to have an exponential solution with an argument that is negative (damping) and imaginary (oscillatory). So we try $\theta = A\exp(at)$. Plugging this into the equation, we have

$$(a^2 + a\gamma + \omega_0^2)Ae^{at} = 0 \,.$$

We divide by $A\exp(at)$ and note that we have a solution provided

$$a^2 + a\gamma + \omega_0^2 = 0 \,.$$

Solving the quadratic equation for a, we find

$$a = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \,.$$

The imaginary part is the oscillation frequency (provided it's real and it is because the statement of the problem said that there was an oscillation!). We read off:

$$\omega_1 = \boxed{\pm\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}} \,.$$